COHERENT AND STOCHASTIC MOTION OF IONS IN TWO OBLIQUE ELECTROSTATIC WAVES

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ABSTRACT

The stochastic motion of ions in the presence of a background magnetic field and electrostatic waves is of interest in both laboratory and space plasmas. Ion heating can be achieved by a single perpendicular or oblique wave driving particles into chaotic dynamics [C. Karney, *Phys. Fluids* 21,9 (1978), G. Smith, A. Kaufman, *Phys. Fluids* 21,12 (1978)].

A spectrum of multiple waves allows for phenomena not possible with a single wave. For instance, ions can be coherently accelerated from low perpendicular energies to the stochastic region by two waves whose frequencies $\omega_1, \omega_2$ differ by an integer multiple of the cyclotron frequency [D. Bénisti, A. K. Ram, A. Bers, *Phys. Plasmas*, 5,9 (1998)]:

$$\omega_1 - \omega_2 = N\omega_{ci}$$

It has recently been shown that two perpendicular waves may explain the high-energy tail of $H^+$ and $O^+$ distributions in the upper ionosphere [A. K. Ram, A. Bers, D. Bénisti, *J. Geophys. Res.*, 103,A5 (1998)].

We show that waves with finite parallel wavenumbers $k_{1z}, k_{2z}$ can also produce coherent acceleration. This occurs provided the parallel wavenumbers are sufficiently close to each other, regardless of how large they are. The resonance condition applies to the Doppler-shifted wave frequencies:

$$\left(\omega_1 - k_{1z}v_z\right) - \left(\omega_2 - k_{2z}v_z\right) = N\omega_{ci}$$

A nonzero $k_{1z} - k_{2z}$ leads to coherent motion in $v_z$ as well as perpendicular energy. A change in $v_z$ leads to a breakage of the resonance condition, and leads to a severe limitation of the coherent motion. This is similar to what happens for frequencies that do not differ by exactly an integer multiple of $\omega_{ci}$.


**OUTLINE**

- **One Perpendicular Wave:** Stochastic region for gyroradius $k_{\perp}\rho \gtrsim \omega/\omega_{ci}$

- $\omega_1 - \omega_2 = N\omega_{ci}$: Resonant Hamiltonian which describes coherent motion

- **Two Perpendicular Waves**
  - Coherent acceleration of low-energy ions to stochastic region
  - Coherent range in $\rho$ scales linearly with wave frequency
  - Period of coherent oscillation scales like $\omega_1^4$
  - Departure from resonance: $\omega_1 - \omega_2 = N\omega_{ci} + \Delta\omega$, "bandwidth" $\Delta\omega$ where coherent motion persists scales like $\omega_1^{-4}$

- **Two Oblique Waves**
  - Stochastic Motion: Lower bound in $\rho$ similar to one-wave case, but upper bound lower; motion in $\rho$ and $v_z$ stochastic
  - $k_{1z} = k_{2z}$: Coherent motion in $\rho$ persists, similar to perpendicular waves
  - $k_{1z} \neq k_{2z}$: $v_z$ evolves coherently
  - $k_{1z} \neq k_{2z}$: Small difference $k_{1z} - k_{2z}$ can severely limit coherent energization: Ion sees Doppler-shifted wave frequencies, which depart from resonance as $v_z$ evolves coherently
Equations of Motion

- Ion moving in two electrostatic waves and uniform $\vec{B}$:

$$M \frac{d^2 \vec{x}}{dt^2} = q \sum_{i=1}^{2} \Phi_i \vec{k}_i \sin(\vec{k}_i \cdot \vec{x} - \omega_i t) + q\vec{v} \times \vec{B}$$

- Normalizations: time to $\omega_{ci} \equiv qB_0/M$, distances to $k_{1x}$.

$$\nu_i \equiv \frac{\omega_i}{\omega_{ci}} \quad \epsilon_i \equiv \left( \frac{\omega_{Bi}}{\omega_{ci}} \right)^2$$

$$\omega_{Bi} \equiv \sqrt{\frac{q k_{1x}^2 \Phi_i}{M}}$$

- Hamiltonian formulation:

$$h(\vec{p}, \vec{x}, t) = \frac{1}{2}(\vec{p} - x\hat{y})^2 + \sum_i \epsilon_i \cos(k_{ix}x + k_{iz}z - \nu_i t)$$

- Gyro-Variables:

$$\rho \equiv \sqrt{v_x^2 + v_y^2} = \text{gyroradius} \quad I \equiv \frac{\rho^2}{2} = \text{perp. energy} \quad \phi \equiv \arctan\left( -\frac{v_y}{v_x} \right) = \text{gyrophase}$$

- Hamiltonian for gyro-variables:

$$H(I, v_z, \phi, z, t) = I + \frac{1}{2}v_z^2 + \sum_i \epsilon_i \cos(k_{ix}\rho \sin \phi + k_{iz}z - \nu_i t)$$
One Perpendicular Wave: Stochastic Region in $\rho$


$$H = I + \epsilon \cos(\rho \sin \phi - \nu t)$$

Surface of Section: $\phi = 2\pi n$

$$\theta \equiv (\nu t) \mod (2\pi)$$

$\nu = 40.37, \epsilon = 4$

- Ions below stochastic region $\rho \gtrsim \nu - \sqrt{\epsilon}$ not energized.
Coherent Motion when $\nu_1 - \nu_2$ an Integer


Second-order resonance: $\nu_1 - \nu_2 \equiv N = \text{integer}$

Initial condition: $v_{z0} = 0$

\[\epsilon_1 = \epsilon_2 = 4 \quad k_{1x} = k_{2x} = 1 \quad k_{1z} = k_{2z} = 0\]

$\nu_1 = 40.37 \quad \nu_2 = 39.37$

$\nu_1 - \nu_2 = \text{integer}$

$\nu_1 = 40.37 \quad \nu_2 = 39.369$

$\nu_1 - \nu_2 \neq \text{integer}$
**Coherent Motion: Lie Perturbation Technique**


\[ x = (p, q) = \text{physical coordinates} \quad \bar{x} = (\bar{p}, \bar{q}) = \text{new coordinates} \]

\[ \bar{H}(\bar{x}) \text{ describes resonant motion in } H(x) \]

Dependence on perturbation parameter \( \epsilon \), ensures \( x \rightarrow \bar{x} \) is canonical:

\[ \frac{\partial \bar{x}}{\partial \epsilon} = [\bar{x}, w(\bar{x}, t)] \quad \bar{x}(\epsilon = 0) = x \]

\[ [f, g]_x = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) = \text{Poisson bracket} \quad w = \text{“Lie generating function”} \]

Coordinate transformation:

\[ (T f)(x) = f(\bar{x}) \quad f(\bar{x}) = \bar{x} \rightarrow Tx = \bar{x} \]

Equation for \( T \):

\[ \frac{\partial T}{\partial \epsilon} f(x) = -T[w(x, t), f(x)]_x \]

Transformed Hamiltonian:

\[ \bar{H} = T_{\epsilon}^{-1} H + T_{\epsilon}^{-1} \int_0^\epsilon d\epsilon' T_{\epsilon'} \frac{\partial w}{\partial t} \]
Deprit Perturbation Series


\[ H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \quad \bar{H} = H_0 + \epsilon \bar{H}_1 + \epsilon^2 \bar{H}_2 \]

Expand Lie generating function:

\[ w = w_1 + \epsilon w_2 \]

Coordinate change \( T \) must be near-identity:

\[ \bar{x} = T x = x - \epsilon [w_1, x] + O(\epsilon^2) \]

To keep \( T \) near-identity, \( w_i \) must remain small.

Expand \( \bar{H} \) equation:

\[
\begin{align*}
D_0 w_1 &= H_1 - H_1 \\
D_0 w_2 &= 2(\bar{H}_2 - H_2) - [w_1, \bar{H}_1 + H_1]
\end{align*}
\]

\[ D_0 f = \frac{\partial f}{\partial t} + [f, H_0] = \text{derivative along unperturbed orbit} \]

Choose \( \bar{H}_i \) to remove secularities in \( w_i \) equation.
Two-Wave Perturbation Theory

\[ H = H_0 + H_1 \quad H_0 = I + \frac{1}{2}v_z^2 \quad H_1 = \sum_{i} \epsilon_i \cos(k_i x \rho \sin \phi + k_i z z - \nu_i t) \]

Unperturbed (\( \epsilon_i = 0 \)) orbits: \( I = \text{const.}, \quad v_z = 0, \quad \phi = t, \quad z = z_0 \).

\( O(\epsilon) \):

\[ (\partial_t + \partial_{\phi} + v_z \partial_z)w_1 = \bar{H}_1 - H_1 = \bar{H}_1 - \sum_{i,m} \epsilon_i J_m(k_i x \rho) \cos(m \phi + k_i z z - \nu_i t) \]

No resonant terms:

\[ \bar{H}_1 = 0 \quad w_1 = -\sum_{i,m} \frac{\epsilon_i J_m(k_i x \rho)}{m - (\nu_i - k_i z v_z)} \sin(m \phi + k_i z z - \nu_i t) \]

\( O(\epsilon^2) \):

\[ (\partial_t + \partial_{\phi} + v_z \partial_z)w_1 = 2\bar{H}_2 - [w_1, H_1] \]

\([w_1, H_1]\) gives terms containing

\[ \cos \left[ (m - n)\phi - (\nu_1 - \nu_2)t + (k_{1z} - k_{2z})z \right] \]

\( \nu_1 - \nu_2 \in \mathbb{Z} \): Resonate along unperturbed orbits \( \rightarrow \) secular growth in \( w_2 \).

\[ \bar{H}_2 = \frac{1}{2} (\text{resonant terms in } [w_1, H_1]) \]
Second-Order Hamiltonian for Coherent Motion

Coherent Hamiltonian: \[ \bar{H}(\bar{I}, \bar{v}_z, \bar{\psi}, \bar{z}) = \frac{1}{2} \bar{v}_z^2 + S_0(\bar{I}, \bar{v}_z) + S_-(\bar{I}, \bar{v}_z) \cos(\bar{N}\bar{\psi} + \Delta k_z \bar{z}) \]

\[ \bar{\psi} = \bar{\phi} - t = \text{angle in rotating gyro-frame} \]

\[ N = \nu_1 - \nu_2 \quad \Delta k_z = k_{1z} - k_{2z} \]

\( S_0, S_- \) are second-order in wave amplitudes: \( S_0 \sim \epsilon_1^2, \epsilon_2^2 \quad S_- \sim \epsilon_1 \epsilon_2 \)

- Barred coordinates differ from physical coordinates by incoherent fluctuations, e.g.:

\[ I = \bar{I} - \epsilon_i \sum_m \frac{mJ_m(k_{ix}\bar{\rho})}{m - \nu_i} \cos(m\bar{\phi} + k_{iz}\bar{z} - \nu_it) + O(\epsilon_i^2) \]

**Coherent Motion in \( v_z \)**

- \( \bar{H} \) is a constant of the motion. Second constant of the motion:

\[ \frac{d}{dt} \left( \frac{\bar{v}_z - \Delta k_z}{N} \bar{I} \right) = 0 \quad \rightarrow \quad \bar{v}_z = v_{z0} + \frac{\Delta k_z}{N}(\bar{I} - I_0) \]
Bounds of coherent motion

- $\bar{v}_z$ is a function of $\bar{I}$:

$$\bar{H} = \frac{1}{2} \bar{v}_z (\bar{I})^2 + S_0 (\bar{I}) + S_- (\bar{I}) \cos(N \bar{\psi} - \Delta k_z \bar{z})$$

$$\cos(N \bar{\psi} - \Delta k_z \bar{z}) = \frac{\bar{H} - \frac{1}{2} \bar{v}_z^2 - S_0}{S_-}$$

$$|\cos x| \leq 1 \quad \rightarrow \quad |\bar{H} - \frac{1}{2} \bar{v}_z^2 - S_0| > |S_-| \quad \text{forbidden}$$

Potential barriers:

$$H_{\pm}(\bar{I}) = \frac{1}{2} \bar{v}_z^2 + S_0 \pm |S_-| \quad H_- \leq \bar{H} \leq H_+$$

Turning points in $\bar{I}$:

$$\bar{H} = \frac{1}{2} \bar{v}_z^2 + S_0 \pm |S_-|$$

Occur when

$$N \bar{\psi} - \Delta k_z \bar{z} = m\pi$$
Expressions for the $S$'s

\[
S_0 = S_{0x} + S_{0z}
\]
\[
S_{0x} = -\frac{1}{2\bar{\rho}} \sum_i k_{ix} \epsilon_i^2 \frac{m}{m - \mu_i} J_{m,i} J'_{m,i}
\]
\[
= \frac{\pi}{8} \sum_i k_{ix}^2 \epsilon_i^2 J_{\mu_i+1,i} J_{-\mu_i-1,i} - J_{\mu_i-1,i} J_{-\mu_i+1,i} \sin \pi \mu_i
\]
\[
S_{0z} = \frac{1}{4} \sum_i k_{iz}^2 \epsilon_i^2 \left(\frac{1}{(m - \mu_i)^2}\right) J_{m,i}^2
\]
\[
= -\frac{\pi}{4} \sum_i k_{iz}^2 \epsilon_i^2 \frac{\partial}{\partial \mu_i} \frac{J_{\mu_i,i} J_{-\mu_i,i}}{\sin \pi \mu_i}
\]

\[
S_\sim = S_{-x} + S_{-z}
\]
\[
S_{-x} = -\frac{1}{4\bar{\rho}} \epsilon_1 \epsilon_2 \left(\frac{1}{m - \mu_1} + \frac{1}{m - \mu_3}\right) \times
\]
\[
\times (k_{1x}(m - N) J'_{m,1} J_{m-N,2} + k_{2x} m J_{m,1} J'_{m-N,2})
\]
\[
S_{-z} = \frac{1}{4} k_{1z} k_{2z} \epsilon_1 \epsilon_2 \left(\frac{1}{(m - \mu_1)^2} + \frac{1}{(m - \mu_3)^2}\right) J_{m,1} J_{m-N,2}
\]

\[
\mu_i = \nu_i - k_{iz} \bar{\nu}_z \quad \text{for} \quad i = 1, 2 \quad \mu_3 = \nu_1 - k_{2z} \bar{\nu}_z \quad J_{m,i} = J_m(k_{ix} \bar{\rho})
\]
Two Perpendicular Waves: Coherent $\rho$ Motion

Full Motion

Coherent Motion

Potential Barriers
Two Perpendicular Waves: Scaling with wave frequencies

\[ \xi \equiv \frac{\bar{\rho}}{\nu_1} \]

- Range of motion in \( \xi \) does not change much with \( \nu_1 \).

Rescaled \( \tilde{H} \) vs. \( \xi \) for \( \nu_1 - \nu_2 = 1 \)

Coherent range of motion in \( \xi \) for \( \xi_0 = 0.4 \)
Period of Coherent Oscillation for Perpendicular Waves

- coherent motion is oscillatory in $\bar{I}$ with a period $\gg$ cyclotron period.

For $\nu_1 \approx \nu_2$ and $\epsilon_1 = \epsilon_2$,

Period scaling:

$$\tau \equiv \frac{2\pi}{N\langle d\psi/dt \rangle} \sim \frac{\nu_1^4}{N\epsilon_1^2}$$

- Increasing wave frequency vastly increases period of coherent motion.

Period for Orbits of $\tilde{H}$ vs. $\nu_1^4$ scaling

lines match $\tau$ at $\nu_1 = 40.37$
\( \nu_1 - \nu_2 \) not an Integer: “Bandwidth” for Coherent Motion

\[ \nu_1 - \nu_2 = N + \Delta\nu \] : resonant terms for \( \Delta\nu = 0 \) are near-resonant

\[ \bar{H} = -\Delta\nu \bar{I} + S_0 + S_\perp \cos N\bar{\psi} \]

\( \Delta\nu_* \) : Critical \( \Delta\nu \) when \( -\Delta\nu \bar{I} \) dominates over \( S_0 \) term and starts limiting motion

\[ \Delta\nu_* \sim \frac{\epsilon_1^2 k_1^2}{\nu_1^4} \]

\( \xi \) Range for \( \frac{\nu_1 = 10.37}{N = 1} \) vs. \( \Delta\nu \)

for \( \xi_0 = 0.4, \phi_0 = \pi/2 \)

Plateau stops at \( \Delta\nu_* = 0.02 \)

\[ 0.02/6 \cdot 10^{-5} = 333 \]

\[ (40.37/10.37)^4 = 230 \]
Oblique Waves with $k_{1z} = k_{2z}$: Coherent Motion in $\rho$

\[ \bar{v}_z = v_{z0} + \frac{\Delta k_z}{N} (\bar{I} - I_0) \]

$\Delta k_z = 0$: no coherent motion in $v_z$

$\rho$ vs. $t$ for $45^\circ$ waves

$v_z$ vs. $t$ for $45^\circ$ waves

$H_\pm$ vs. $\bar{\rho}$ for $k_{1z} = 0, 0.1, 1$

Range of $\xi$ Motion vs. $k_{1z}$ for $\xi_0 = 0.4$
Oblique Waves with $k_{1z} \neq k_{2z}$ : Coherent Motion Limited

$\bar{v}_z = v_{z0} + \frac{\Delta k_z}{N}(\bar{I} - I_0)$

$\Delta k_z \neq 0$ : small coherent motion in $v_z$

Second-Order Constant $\bar{v}_z = \frac{\Delta k_z}{N} \bar{I}$

Variation $< 0.3\%$
$k_{1z} \neq k_{2z}$: Potential Barriers Pulled Closer

$$\tilde{H} = \frac{1}{2} \frac{\Delta k_z^2}{N^2} (\bar{I} - \bar{I}_0)^2 + S_0 + S_- \cos(N \bar{\psi} - \Delta k_z \bar{z})$$

$\Delta k_{z*}$: Critical $\Delta k_z$ when motion limited

$$\Delta k_{z*} \sim \frac{N \epsilon_1 k_{1x}}{v_1^3}$$

As $\Delta k_z$ increases, a smaller change in $\bar{I}$ moves ions between the potential barriers $H_-$ and $H_+$

$\xi$ range $\Delta k_z$ for $\nu_1 = 10.37$ , $N = 1$, $\xi_0 = 0.4$

$\xi$ range $\Delta k_z$ for $\nu_1 = 40.37$ , $N = 1$, $\xi_0 = 0.4$
\( k_{1z} \neq k_{2z} : \) Departure from \( \nu_1 - \nu_2 = \text{integer} \)

![Diagram showing wave frequency and ion velocity in LAB and ion z-frames]

Ion sees Doppler-shifted frequencies: \( \nu_i = \nu_{i0} - k_{iz}v_z \)

Resonance condition: \( \nu_1 - \nu_2 = \nu_{10} - \nu_{20} - \Delta k_z v_z = N \)

- \( \Delta k_z \neq 0 : v_z \) changes coherently, resonance condition cannot stay satisfied.

### Heating a Distribution in \( v_z \)

Lab frame: \( v_z(t = 0) = v_{z0} \)

Resonance condition: \( \nu_{10} - \nu_{20} - \Delta k_z v_{z0} = N \)

- \( \Delta k_z = 0 : \) Resonance for \( \nu_{10} - \nu_{20} = N \). All ions resonate. Ions with larger \( k_{iz}v_{z0} \) see smaller \( \nu_i \).

- \( \Delta k_z \neq 0 : \) Only certain \( v_{z0} \) resonate: \( v_{z0N} = \frac{\nu_{10} - \nu_{20} - N}{\Delta k_z} \quad N = 1, 2, 3, \ldots \)
CONCLUSIONS

• Coherent Energization to stochastic region possible for oblique waves provided \((k_{1z} - k_{2z})\) is small
• \(k_{1z} \neq k_{2z}\) leads to coherent motion in \(v_z\), which Doppler shifts waves away from resonance
• Lower wave frequencies \(\omega_1, \omega_2\) are “more favorable:”
  – Departure of \((\omega_1 - \omega_2)/\omega_{ci}\) from an integer, and \(k_{1z}\) from \(k_{2z}\), that still permit coherent energization scale like \(\omega_1^{-4}\) and \(\omega_1^{-3}\), respectively
  – Period of coherent oscillation scales like \(\omega_1^4\)

Questions? Comments? Reprints?