STOCHASTIC PARTICLE MOTION DUE TO MULTIPLE ELECTROSTATIC WAVES

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ABSTRACT

Stochastic motion of particles in the presence of a background magnetic field and electrostatic waves is of interest in both laboratory and space plasmas. Ion heating in fusion experiments can be achieved by a single wave driving particles into chaotic dynamics. Wave mechanisms are also believed to account for the energization of ions from the ionosphere to the magnetosphere. In particular, it has recently been shown that two perpendicular waves may explain the high-energy tail of H^+ and O^+ distributions in the upper ionosphere (Ram *et al*, *J. Geophys. Res.*, 103:A5, 1998).

In a uniform magnetic field $\vec{B}_0$, one perpendicular wave causes particles within a range of perpendicular energies to move stochastically when the wave amplitude exceeds a threshold (Karney, *Phys. Fluids* 21(9), 1978). Two perpendicular waves whose frequencies differ by an integer multiple of the cyclotron frequency ($\omega_1 - \omega_2 = N\Omega$) can coherently accelerate particles from low energies into the one-wave chaotic regime. We show that oblique waves can also produce coherent acceleration, provided their parallel wavenumbers are sufficiently close. The resonance condition now applies to the Doppler-shifted wave frequencies: $\omega_1 - \omega_2 - v_z(k_{1z} - k_{2z}) = N\Omega$. The coherent evolution of v_z is proportional to $k_{1z} - k_{2z}$, so that N varies over time and does not stay an integer when the k_z's differ. This effect does not qualitatively alter the coherent motion when the variation in N is small.

Poster Outline

- Stochasticity with one perpendicular wave
- Coherent acceleration with two perpendicular waves into stochastic region
- Multiple timescale analysis for coherent acceleration
- Effects of finite $k_{\|}$ and parallel dynamics
- Phase dependence of coherent acceleration (including relative angle perpendicular to $\vec{B}_0$)

Particle Equation of Motion

- Particle moving in uniform background $\vec{B}_0 = B_0\hat{z}$ and fixed electrostatic waves (no feedback of particles on waves):

$$m\ddot{\vec{x}} = q\sum_{i=1}^{2} E_i\hat{k}_i \sin(\vec{k}_i \cdot \vec{x} - \omega_i t) + q\vec{v} \times \vec{B}_0$$

- Nondimensionalize time to $\Omega \equiv qB_0/m$, length to typical k^{-1}:

$$\ddot{\vec{x}} = \sum_i \vec{\epsilon_i} \sin(\vec{k}_i \cdot \vec{x} - \nu_i t) + \vec{v} \times \hat{z}$$

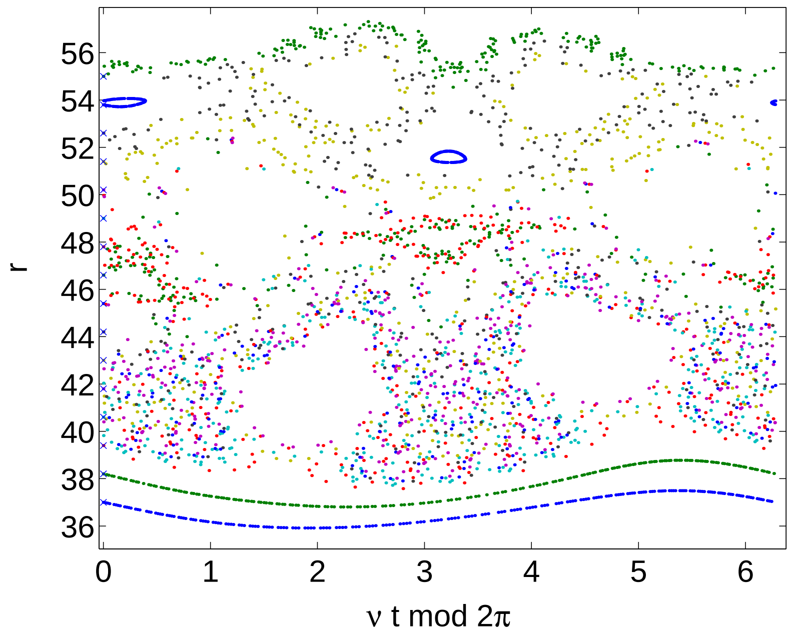
$$\vec{\epsilon_i} \equiv \frac{qkE_i}{m\Omega^2}\hat{k}_i = \left(\frac{\omega_B}{\Omega}\right)^2 \hat{k}_i, \qquad \nu_i = \frac{\omega_i}{\Omega}$$

$\vec{k}_1 = (k_{1\perp}, 0, k_{1z}), \quad \vec{k}_2 = (k_{2\perp}\cos\alpha, k_{2\perp}\sin\alpha, k_{2z})$, similar for $\vec{\epsilon}$

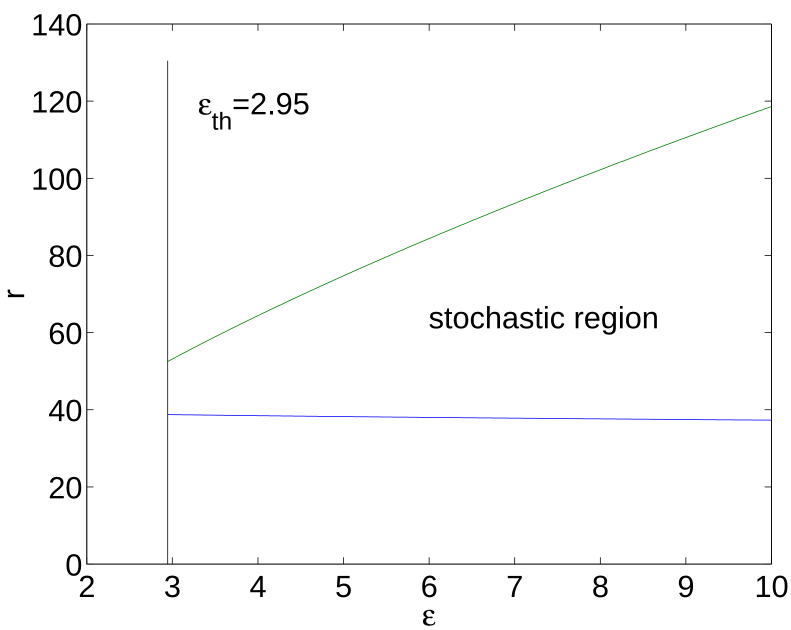
One Perpendicular Wave: Stochastic Motion

- Larmor radius $r \equiv \sqrt{v_x^2 + v_y^2}/\Omega$, gyrophase $\phi \equiv \arctan(-v_y/v_x)$.
- Perpendicular dynamics chaotic for $\epsilon > \epsilon_{th} \equiv \nu^{2/3}/4$ and $\nu - \sqrt{\epsilon} < r < (2/\pi)^{1/3}(4\epsilon\nu)^{2/3}$ (Karney).

$\nu = 40.47, \vec{k}_1 = \hat{x}, \epsilon_1 = 3.2, \epsilon_2 = 0.$ Surface of Section(ϕ=0)



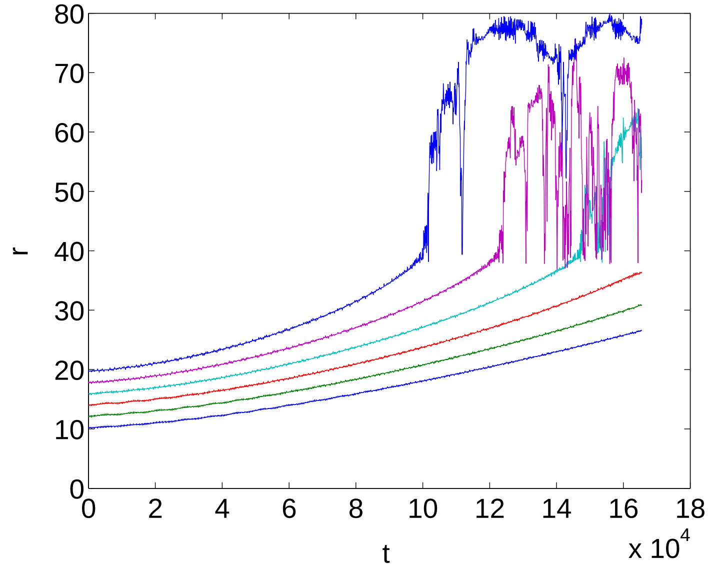
Stochasticity Threshold



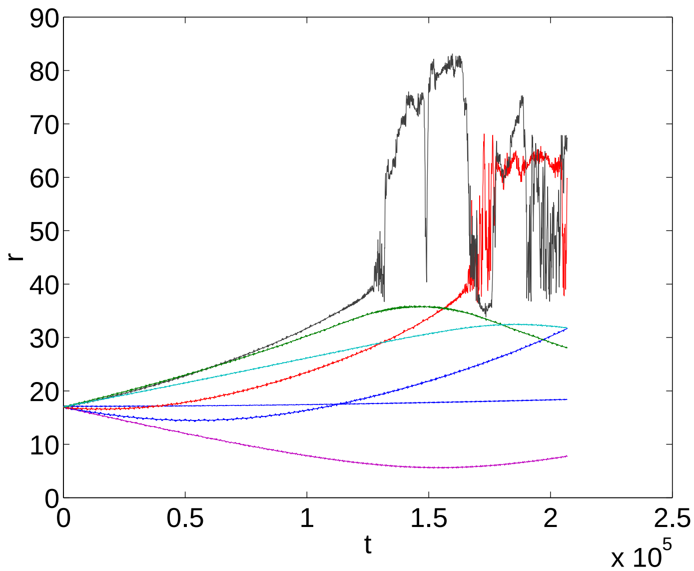
Two Perpendicular Waves: Coherent Acceleration

- Two perpendicular waves ($\vec{k}_i = k_i\hat{x}$) can coherently accelerate or decelerate particles and change their perpendicular energy.
- Particles in regular phase space for one wave can reach stochastic region (same lower bound as one wave with $\epsilon = \max(\epsilon_1, \epsilon_2)$, $\nu = \min(\nu_1, \nu_2)$).
- Resonance with frequency *difference*:
  $\nu_1, \nu_2 \notin \mathcal{Z}$, but $N \equiv \nu_1 - \nu_2 \in \mathcal{Z}$

$\nu_1 = 40.47, \nu_2 = \nu_1 - 1, \epsilon_1 = \epsilon_2 = 3.2, \vec{k_1} = \vec{k_2} = \hat{x}$



Energization Depends on Phase



Oblique Waves

- Resonance condition is that the difference of the z Doppler-shifted frequencies is an integer:

$$N \rightarrow \nu_1 - v_z k_{1z} - (\nu_2 - v_z k_{2z}) = \nu_1 - \nu_2 - v_z(k_{1z} - k_{2z}) \in \mathcal{Z}$$

- Analysis strictly holds only for $N(t)$ =constant

Multiple Timescale Analysis

- Multiple timescale analysis yields equations with slow, second-order evolution.

  $$\delta \sim \frac{\Omega}{\omega} \qquad T_i = \delta^i t \qquad x(\vec{t}) = \vec{x}_0(T_0, T_1, ...) + \delta \vec{x}_1(T_0, T_1, ...) + ...$$

  We order typical $\epsilon_i \sim \delta$ (waves enter at first order).

- Timescales: $T_0 = \Omega^{-1}$, $\qquad T_1 = \omega^{-1}$, $\qquad T_2 =$ nonlinear timescale

- $O(\delta^0) : \vec{x_0} = (r \sin(T_0 + \phi), r \cos(T_0 + \phi), v_{z0} T_0) + \vec{C_0}$

Slow evolution of r, ϕ, v_{z0} chosen to prevent higher-order $\vec{x}_i$'s from growing faster than $\vec{x}_0$.

- $O(\delta) : r, \phi$ independent of T_1; get small-scale oscillations.

- $O(\delta^2) : \vec{x}_1 \cos(\vec{k}_i \cdot \vec{x}_0 - \nu_i t)$ terms give resonant driving terms, nontrivial equations for r, ϕ.

Slow (Second-Order) Evolution of r, ϕ

- New constant of motion, "energy" H, exists to second order.
- Action $I \equiv r^2/2$ conjugate to ϕ, construct Hamiltonian:

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial I}, \qquad \frac{dI}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H \equiv S_{1\perp}(I) + S_{1z}(I) + \cos N(\phi + \alpha) S_{2a}(I) + \sin N(\phi + \alpha) S_{2b}(I)$$

$$\Rightarrow H = S_{1\perp} + S_{1z} + \cos \beta(I, \phi) \sqrt{S_{2a}^2 + S_{2b}^2}$$

$$\beta \equiv N(\phi + \alpha) + \arctan \frac{S_{2b}}{S_{2a}}$$

- Allows bounds to be put on H:

$$H_{\pm} = S_{1\perp} + S_{1z} \pm \sqrt{S_{2a}^2 + S_{2b}^2}$$

$$H_{-} \leq H \leq H_{+}$$

- Gives rise to ϕ dependence of acceleration.

Expressions for S's

- $J_{im} \equiv J_m(k_{i\perp} r)$ = Bessel function, $\nu \equiv \nu_1 - k_{1z} v_z$
  sum over $m = -\infty : \infty$

$$S_{1\perp} = -\frac{1}{4}\frac{\epsilon_{1\perp}^2 J_{1m}^2 + \epsilon_{2\perp}^2 J_{2m-N}^2}{1-(m-\nu)^2} \qquad S_{1z} = \frac{1}{4}\frac{\epsilon_{1z}^2 J_{1m}^2 + \epsilon_{2z}^2 J_{2m-N}^2}{(m-\nu)^2}$$

$$\begin{aligned} S_{2a} &= -\frac{\epsilon_{1\perp}\epsilon_{2\perp}}{2}\left(\cos\alpha\cos(m\alpha) + \frac{\sin\alpha\sin(m\alpha)}{m-\nu}\right)\frac{J_{1m}J_{2m-N}}{1-(m-\nu)^2} \\ &\quad +\frac{\epsilon_{1z}\epsilon_{2z}}{2}\cos(m\alpha)\frac{J_{1m}J_{2m-N}}{(m-\nu)^2} \end{aligned}$$

$$\begin{aligned} S_{2b} &= \frac{\epsilon_{1\perp}\epsilon_{2\perp}}{2}\left(\cos\alpha\sin(m\alpha) + \frac{\sin\alpha\cos(m\alpha)}{m-\nu}\right)\frac{J_{1m}J_{2m-N}}{1-(m-\nu)^2} \\ &\quad -\frac{\epsilon_{1z}\epsilon_{2z}}{2}\sin(m\alpha)\frac{J_{1m}J_{2m-N}}{(m-\nu)^2} \end{aligned}$$

- Special case: $\alpha = 0$, waves in $x - z$ plane:

$$S_{2a} = -\frac{\epsilon_{1\perp}\epsilon_{2\perp}}{2}\frac{J_{1m}J_{2m-N}}{1-(m-\nu)^2} + \frac{\epsilon_{1z}\epsilon_{2z}}{2}\frac{J_{1m}J_{2m-N}}{(m-\nu)^2}$$

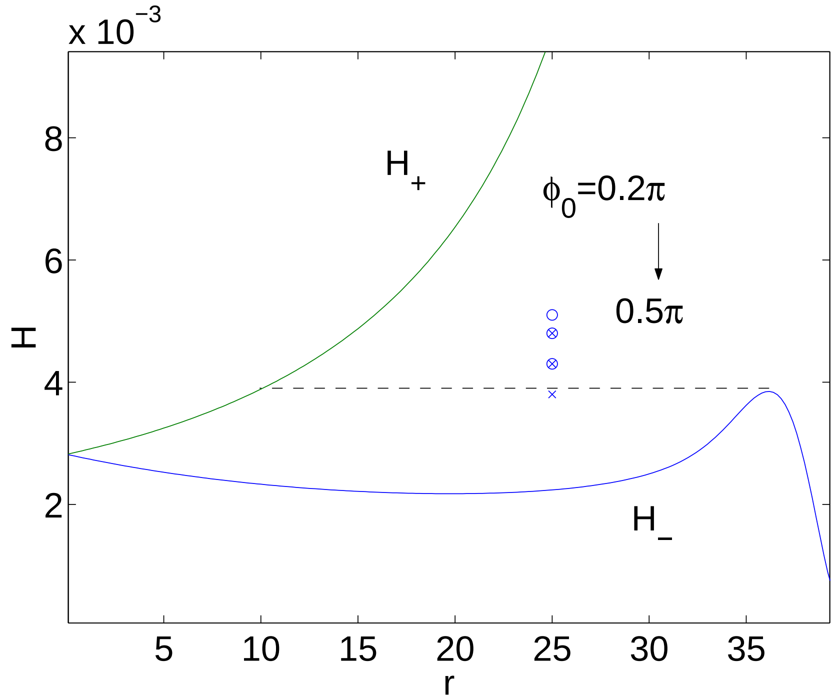
$$S_{2b} = 0$$

$$S_{1\perp} + S_{1z} - |S_{2a}| \le H \le S_{1\perp} + S_{1z} + |S_{2a}|$$

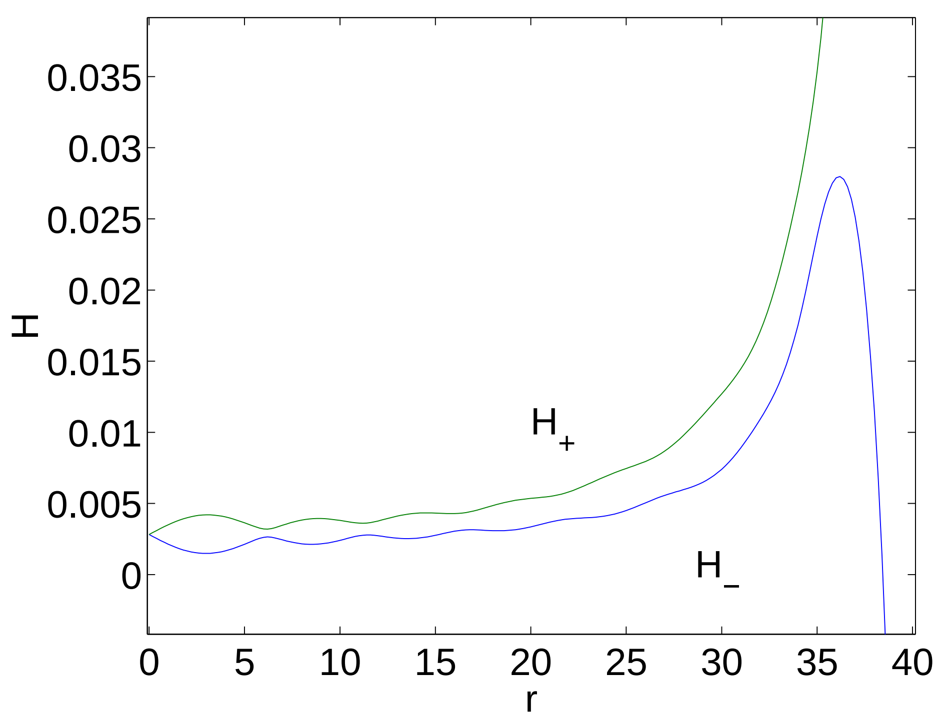
ϕ Dependence of acceleration

- $H_-(r) \leq H(r,\phi) \leq H_+(r)$ puts bounds on r motion

$$\nu_1 = 40.47, \nu_2 = nu_1 + 1, k_{1z} = 0.01, k_{2z} = 0.0101, \epsilon_1 = \epsilon_2 = 3$$



$\alpha = 0.2\pi$ strongly restricts motion



Parallel Dynamics

- Parallel dynamics linked to I via a constant (to second order) of the motion:

$$K \equiv v_{z0} - \frac{k_{1z} - k_{2z}}{N} I = \text{const}$$

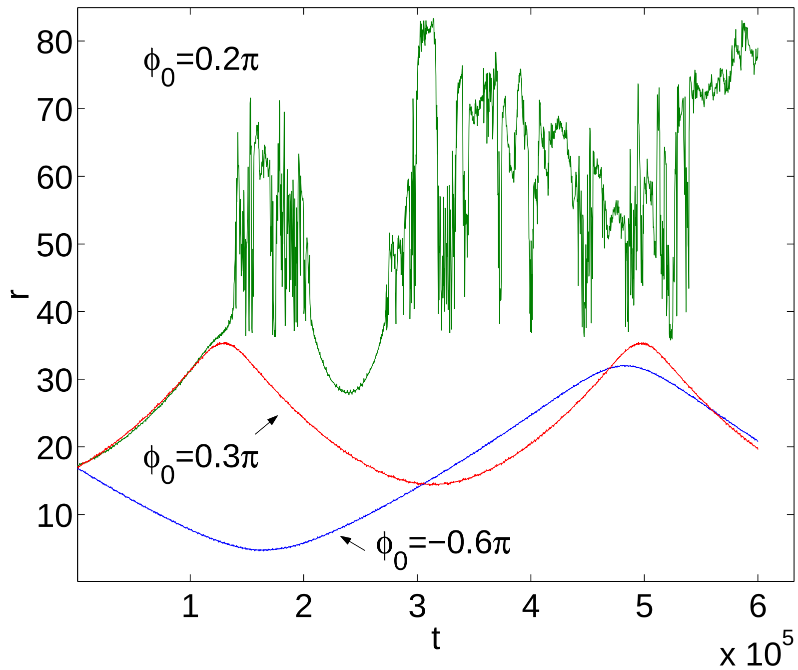
- v_{z0} evolution proportional to *difference* between parallel wavenumbers $k_{1z} - k_{2z}$.
- Slow change in v_{z0} destroys resonance condition by moving N away from an integer.

$$\begin{aligned} k_{1z} \neq k_{2z} \;&\Rightarrow\; N, v_{z0} \quad \text{change over time} \\ &\Rightarrow\; \text{not exact resonance} \end{aligned}$$

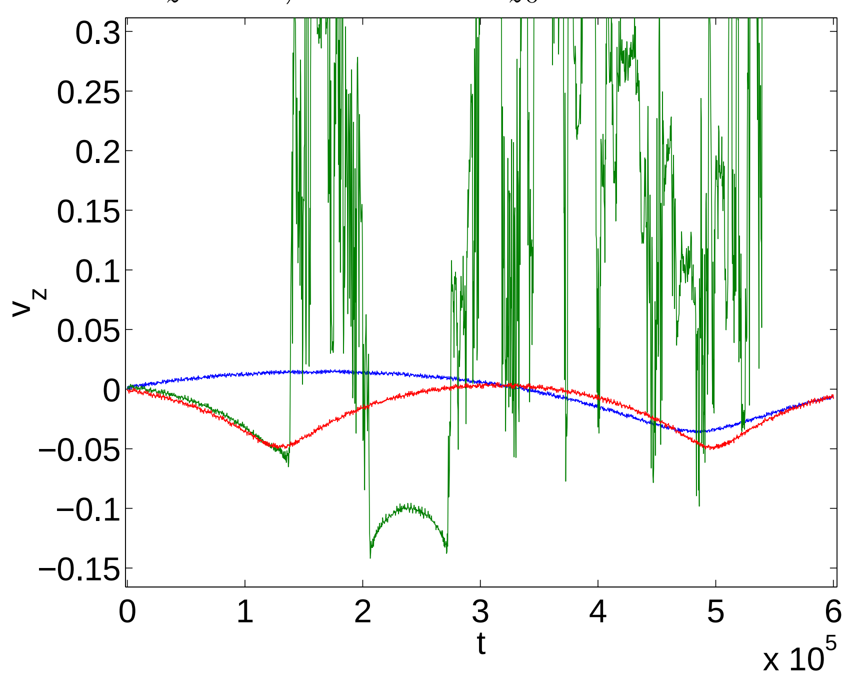
$k_{1z} = 0.01, k_{2z} = 0.0101 : \approx$ **Resonance**

- Slow Evolution is Large-Amplitude Oscillation
- Amplitude, frequency of slow dynamics depend on initial ϕ

r vs. t for same $r(t=0)$, different $\phi(0)$

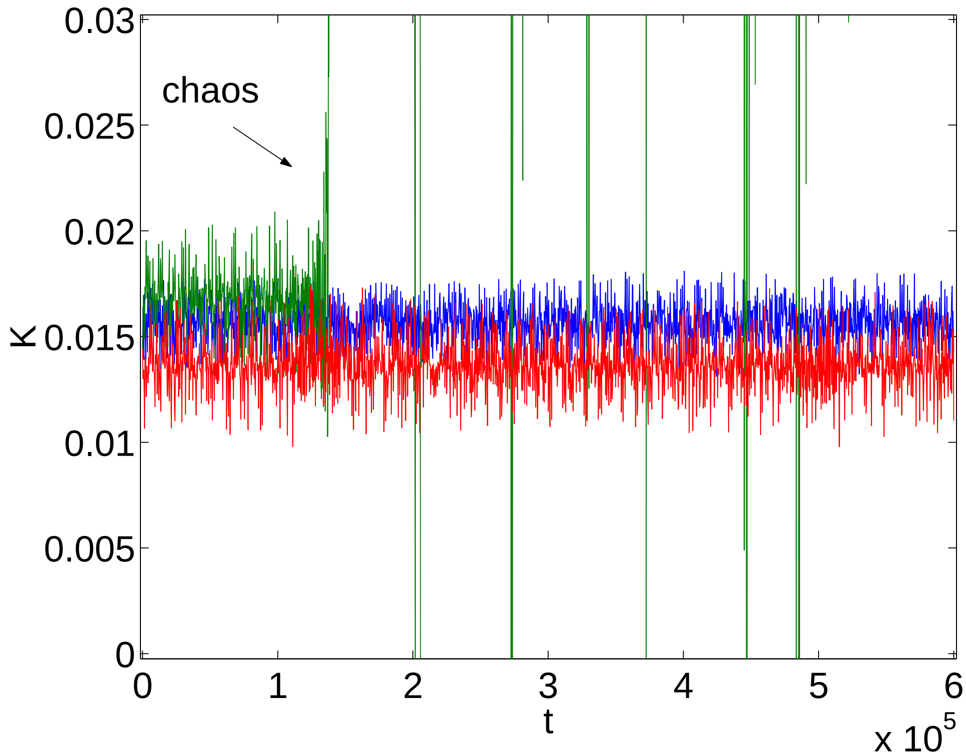


v_z vs. t, $\quad K = v_{z0} + 10^{-4} I$



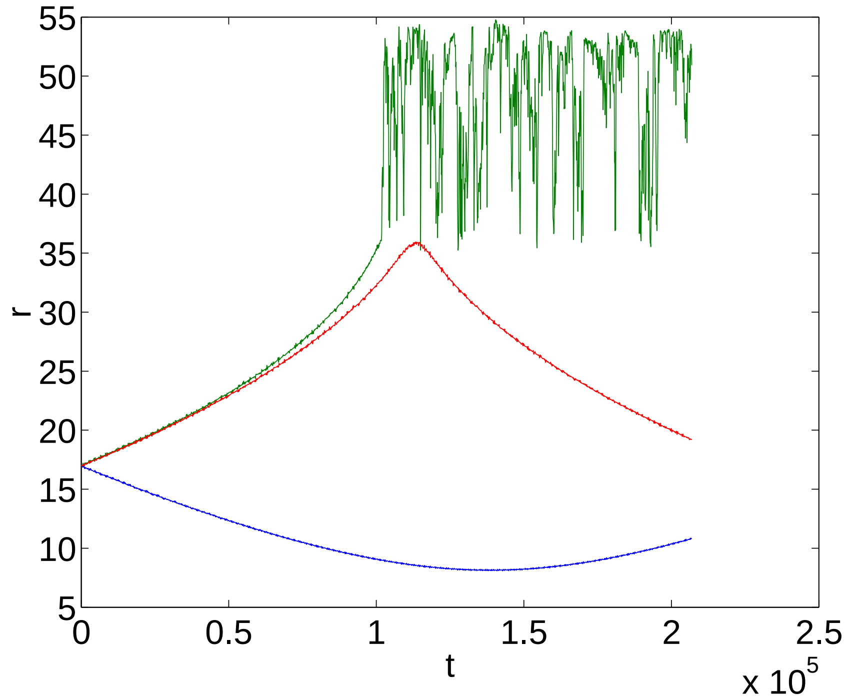
“Constancy” (no secular change) of K

K vs. $t, k_{1z} = 0.01, k_{2z} = 0.0101$

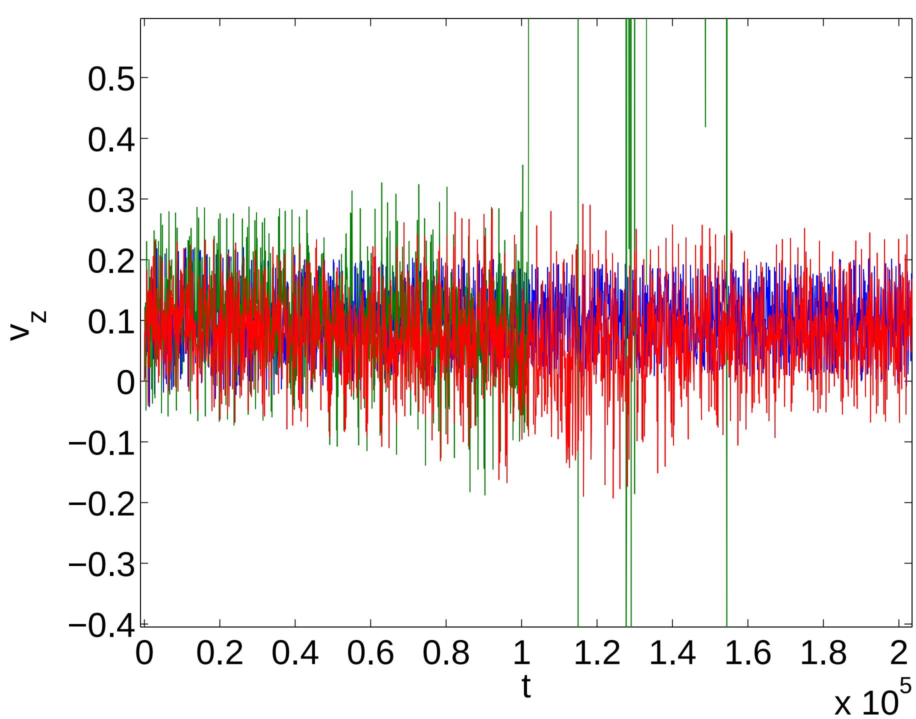


Large k_z's Accelerate if Difference is Small

$k_{1z} = 1, k_{2z} = 1.0001, r$ vs. t



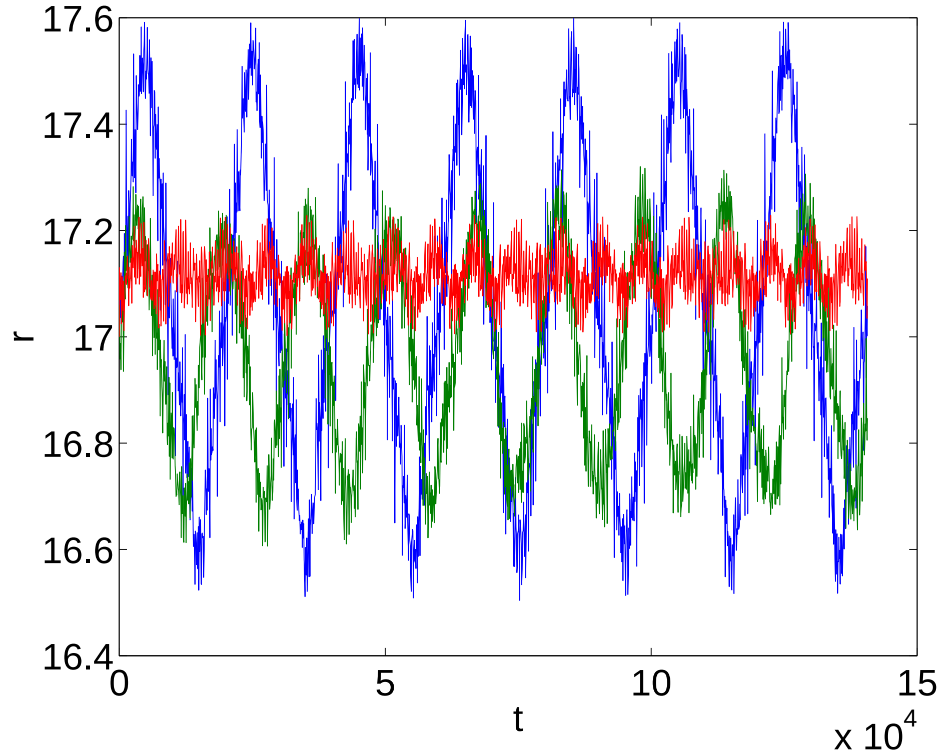
v_z vs. t, $\max(v_z) \approx 80$



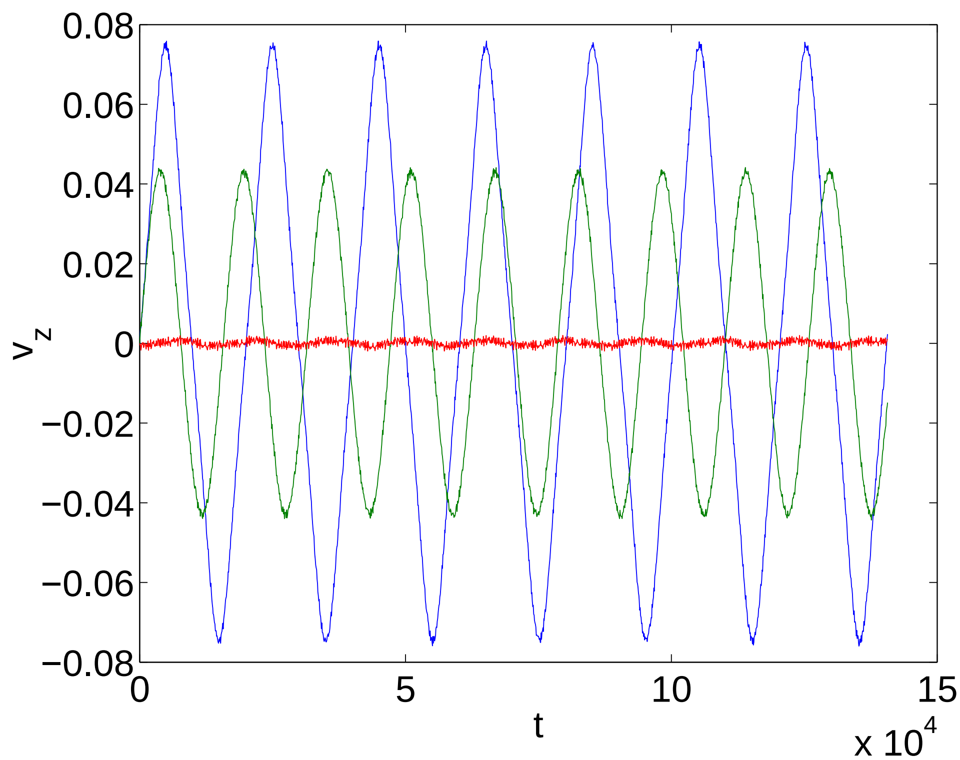
- Acceleration in z masked by first-order oscillations.

$\underline{k_{1z} = 0.01, k_{2z} = 0 : \textbf{Resonance Limited}}$

r vs. t



v_z vs. t



SUMMARY AND FUTURE WORK

- Coherent acceleration by two perpendicular waves can be limited by $k_\parallel$.
- Multiple timescale analysis generalized to include parallel dynamics: second-order constant K.
- Resonance condition modified to $N = \nu_1 - \nu_2 - v_{z0}(k_{1z} - k_{2z})$.
- For small $k_{1z} - k_{2z}$, coherent acceleration persists.
- Phase dependence of acceleration explained by $H_\pm$ bounds, but not accurate when k_z's differ.
- Synergism between perpendicular and oblique one-wave stochasticity with multiple waves?
- Evolution of distribution function? with broadband waves?

Questions? Comments? Reprint? Please leave e-mail